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Super-Poincaré covariant quantization of the superstring

Nathan Berkovits

Instituto de Física Teórica, Universidade Estadual Paulista
Rua Pamplona 145, 01405-900, São Paulo, SP, Brasil
E-mail: nberkovi@ift.unesp.br

Abstract: Using pure spinors, the superstring is covariantly quantized. For the first time, massless vertex operators are constructed and scattering amplitudes are computed in a manifestly ten-dimensional super-Poincaré covariant manner. Quantizable non-linear sigma model actions are constructed for the superstring in curved backgrounds, including the $AdS_5 \times S^5$ background with Ramond-Ramond flux.

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Contents

1. Introduction **1**

2. Pure spinors and Lorentz currents **3**

3. Physical vertex operators **6**

4. Computation of scattering amplitudes **8**

5. Superstring action in a curved background **10**
5.1 Superstring action in $AdS_5 \times S^5$ background 12

6. Concluding remarks **14**

1. Introduction

There are many motivations for covariantly quantizing the superstring. As in any theory, it is desirable to make all physical symmetries manifest in order to reduce the amount of calculations and simplify any cancellations coming from the symmetry. Recently, an additional motivation has come from the desire to construct a quantizable sigma model action for the superstring in curved backgrounds with Ramond-Ramond flux.

Most attempts to covariantly quantize the superstring have started from the classical super-Poincaré invariant version of the Green-Schwarz (GS) action [1]. One quantization approach is based on gauge-fixing the fermionic symmetries to get to "semi-light-cone" gauge where $(\gamma^+\theta)_\alpha = 0$ and $\gamma^+ = \gamma^0 + \gamma^9$ [2]. In this gauge, the covariant Green-Schwarz action simplifies to $S = \int d^2z[\partial x^m \overline{\partial} x_m + \partial x^+(\theta\gamma^-\overline{\partial}\theta)]$. However, even this simplified action cannot be easily quantized since the propagator for θ involves $(\partial x^+)^{-1}$ which is not well-defined.[1] For this reason, it has not yet been possible to use this approach to construct physical vertex operators or compute scattering amplitudes, except in the $p^+ \to 0$ limit that reproduces the light-cone gauge computations [3]. Another approach to quantizing the covariant Green-Schwarz action is based on replacing the fermionic second-class constraints with an appropriate set of first-class constraints [4], sometimes using $\mathrm{SO}(9,1)/\mathrm{SO}(8)$ harmonic variables [5]

[1]On a genus g worldsheet with N punctures, ∂x^+ vanishes at $2g+N-2$ points on the worldsheet. This fact is related to the need for interaction-point operators in the light-cone GS superstring.

which covariantize the semi-light-cone gauge choice. However, despite numerous attempts [6], noone was able to find an appropriate set of first-class constraints which allows the covariant computation of scattering amplitudes.

In the absence of Ramond states, it is possible to quantize the superstring in a manifestly Lorentz-covariant manner using the standard Ramond-Neveu-Schwarz (RNS) formalism. However, none of the spacetime supersymmetries are manifest in the RNS formalism and, in order to explicitly construct the spin field for Ramond states, manifest $\mathrm{SO}(9,1)$ Lorentz invariance must be broken (after Wick-rotation) to a $\mathrm{U}(5)$ subgroup [7]. Recently, an alternative formalism for the superstring was constructed which manifestly preserves this same $\mathrm{U}(5)$ subgroup in addition to manifestly preserving six of the sixteen spacetime supersymmetries [8]. The worldsheet variables of this supersymmetric $\mathrm{U}(5)$ formalism are related to those of the RNS formalism by a field redefinition, allowing one to prove that physical vertex operators and scattering amplitudes in the two formalisms are equivalent. However, the lack of manifest Lorentz invariance makes it difficult to use this formalism to describe the superstring in curved (Wick-rotated) backgrounds which do not preserve $\mathrm{U}(5)$ holonomy.

In this paper, a new formalism for the superstring will be presented which can be quantized in a manifestly super-Poincaré covariant manner. The worldsheet variables of this formalism will consist of the usual ten-dimensional superspace variables in addition to a bosonic spacetime spinor λ^α satisfying the 'pure' spinor condition

$$\lambda^\alpha \gamma^m_{\alpha\beta} \lambda^\beta = 0\,, \tag{1.1}$$

for $m = 0$ to 9. λ^α must be complex to satisfy (1.1) and, after Wick-rotating $\mathrm{SO}(9,1)$ to $\mathrm{SO}(10)$, can be parameterized by eleven complex variables. One of these eleven variables is an overall scale factor, and the other ten parameterize the coset space $\mathrm{SO}(10)/\,\mathrm{U}(5)$. So this new formalism is probably related to a covariantization of the $\mathrm{U}(5)$ formalism of [8]. Although the precise relation between the two formalisms is still unclear, it will be argued in section 2 that pure spinor variables are necessary for equating RNS vertex operators with the GS vertex operators proposed in [4].

In section 3, physical states will be defined as elements in the cohomology of the BRST-like operator

$$Q = \oint dz \lambda^\alpha d_\alpha\,, \tag{1.2}$$

where d_α is the generator of supersymmetric derivatives as defined in [4]. Since $d_\alpha(y) d_\beta(z) \to 2(y-z)^{-1} \Pi_m(z) \gamma^m_{\alpha\beta}$ where Π_m is the supersymmetric translation generator, (1.1) implies that $Q^2 = 0$. Note that the operator of (1.2) was used in [9] by Howe to show that the constraints of ten-dimensional super-Yang-Mills and supergravity can be understood as integrability conditions on pure spinor lines.

Using this definition of physical states, one can easily construct the physical massless vertex operators. For the open superstring, the massless vertex operator in unintegrated form is $V = \lambda^\alpha A_\alpha(x,\theta)$ and in integrated form is

$$V = \int dz(\Pi^m A_m + \partial\theta^\alpha A_\alpha + d_\alpha W^\alpha + N^{mn} F_{mn})\,, \tag{1.3}$$

where A_M are the super-Yang-Mills prepotentials, W^α and F_{mn} are the gauge-invariant superfields whose lowest components are the gluino and the gluon field strength, and N^{mn} is the pure spinor contribution to the Lorentz generator. Except for the N^{mn} term, the vertex operator of (1.3) is that proposed by Siegel in [4]. As will be shown in section 4, these vertex operators can be used to compute scattering amplitudes in a manifestly super-Poincaré covariant manner.

The physical vertex operators for the closed superstring can be obtained by taking the 'left-right' product of two open superstring vertex operators. In section 5, the integrated form of the closed superstring massless vertex operator will be used to construct a quantizable sigma model action for the superstring in a curved superspace background. As a special case, a quantizable sigma model action will be constructed for the superstring in an $AdS_5 \times S^5$ background with Ramond-Ramond flux. This action differs from that of Metsaev and Tseytlin [10] in containing a kinetic term for the fermions which allows quantization.

In section 6, further evidence will be given for equivalence with the RNS formalism and some possible applications of the new formalism will be discussed.

2. Pure spinors and Lorentz currents

In conformal gauge, the left-moving contribution to the covariant Green-Schwarz superstring action can be written as

$$S = \int d^2z \left(\frac{1}{2}\partial x^m \overline{\partial} x_m + p_\alpha \overline{\partial}\theta^\alpha\right),$$

where p_α is related to x^m and θ^α by the constraint $d_\alpha = 0$ with[4]

$$d_\alpha = p_\alpha + \gamma^m_{\alpha\beta}\partial x_m \theta^\beta + \frac{1}{2}\gamma^m_{\alpha\beta}\gamma_{m\,\gamma\delta}\theta^\beta\theta^\gamma\partial\theta^\delta\,. \tag{2.1}$$

Since $d_\alpha(y)d_\beta(z) \to 2(y-z)^{-1}\gamma^m_{\alpha\beta}\Pi_m(z)$ where $\Pi^m = \partial x^m + \theta^\alpha\gamma^m_{\alpha\beta}\partial\theta^\beta$, $d_\alpha = 0$ involves first and second-class constraints. The idea of [4] is to find an appropriate set of first-class constraints constructed from d_α which can replace the second-class constraints. In such a framework, p_α is treated as an independent field and physical vertex operators are annihilated by the first-class constraints. Although an appropriate set of first-class constraints were not found in [4], Siegel used supersymmetry arguments to conjecture that the massless open superstring vertex operator should have the form

$$V = \int dz(\Pi^m A_m + \partial\theta^\alpha A_\alpha + d_\alpha W^\alpha)\,, \tag{2.2}$$

where A_M are the super-Yang-Mills prepotentials and W^α is the super-Yang-Mills spinor field strength.

JHEP04(2000)018

For a gluon, the vertex operator of (2.2) reduces to $V = \int dz(\partial x^m \mathcal{A}_m(x) + \frac{1}{2}(p\gamma^{mn}\theta)\mathcal{F}_{mn}(x))$ where $\mathcal{A}_m$ and $\mathcal{F}_{mn}$ are the ordinary θ-independent gluon gauge field and field strength, which closely resembles the gluon vertex operator in the RNS formalism $V = \int dz(\partial x^m \mathcal{A}_m + \psi^m\psi^n \mathcal{F}_{mn})$. However, there is a crucial difference between the OPE's of the SO(9,1) Lorentz currents $M^{mn} = \frac{1}{2}p\gamma^{mn}\theta$ and $\widehat{M}^{mn} = \psi^m\psi^n$ which will force the introduction of pure spinors. Namely, the OPE of M^{kl} with M^{mn} has a double pole proportional to $\frac{16}{4}(\eta^{kn}\eta^{lm} - \eta^{km}\eta^{ln})$ where the factor of 16 comes from the spinor dimension. However, the double pole in the OPE of $\widehat{M}^{kl}$ with $\widehat{M}^{mn}$ is proportional to $(\eta^{kn}\eta^{lm} - \eta^{km}\eta^{ln})$ without the factor of 16/4. So the vertex operator of [4] can only be equivalent at the quantum level to the RNS vertex operator if one adds a new term to the Lorentz current $M^{mn} = \frac{1}{2}p\gamma^{mn}\theta + N^{mn}$ where N^{mn} satisfies the OPE[2]

$$N^{kl}(y)N^{mn}(z) \to \frac{\eta^{m[l}N^{k]n}(z) - \eta^{n[l}N^{k]m}(z)}{y-z} - 3\frac{\eta^{kn}\eta^{lm} - \eta^{km}\eta^{ln}}{(y-z)^2}\,. \tag{2.3}$$

As will now be shown, such a Lorentz current N^{mn} can be explicitly constructed from a pure spinor λ^α, i.e. a complex bosonic spinor satisfying (1.1). To parameterize the eleven independent complex degrees of freedom of λ^α, it is convenient to Wick-rotate and temporarily break SO(10) to SU(5) $\times$ U(1) as in [8]. The sixteen complex components of λ^α split into $(\lambda^+, \lambda_{ab}, \lambda^a)$ for $a, b = 1$ to 5, which transform respectively as $(1_{5/2}, \overline{10}_{1/2}, 5_{-3/2})$ representations of SU(5) $\times$ U(1) where the subscript denotes the U(1) charge. In terms of the eleven independent complex variables (γ, u_{ab}) transforming as $(1_5, \overline{10}_{-2})$ representations, one can check that

$$\lambda^+ = \gamma\,, \qquad \lambda_{ab} = \gamma u_{ab}\,, \qquad \lambda^a = -\frac{1}{8}\gamma\epsilon^{abcde}u_{bc}u_{de} \tag{2.4}$$

satisfies the pure spinor condition of (1.1). Note that γ-matrices in U(5) notation satisfy $\lambda\gamma^a\lambda = \lambda^+\lambda^a + \frac{1}{8}\epsilon^{abcde}\lambda_{bc}\lambda_{de}$ and $\lambda\gamma_a\lambda = \lambda_{ab}\lambda^b$ where the SO(10) vector has been split into a 5_1 and $\overline{5}_{-1}$ representation.

[2] In four dimensions, $\frac{1}{2}p\gamma^{mn}\theta$ has a double pole proportional to $\frac{4}{4}(\eta^{kn}\eta^{lm} - \eta^{km}\eta^{ln})$, so there is no need to add new Lorentz degrees of freedom when quantizing the four-dimensional superstring [11]. In six dimensions, $\frac{1}{2}p_j\gamma^{mn}\theta^j$ (where $j = 1$ to 2 is an internal SU(2) index) has a double pole proportional to $\frac{8}{4}(\eta^{kn}\eta^{lm} - \eta^{km}\eta^{ln})$, so one needs to add degrees of freedom whose Lorentz current has a double pole with itself proportional to $-(\eta^{kn}\eta^{lm} - \eta^{km}\eta^{ln})$. These degrees of freedom are a bosonic spinor u^α and its conjugate momentum v_α for $\alpha = 1$ to 4. They are the ghosts for the 'harmonic' constraints $\widetilde{d}_\alpha = d_{\alpha 2} - e^{-\rho - i\sigma}d_{\alpha 1}$ of [12] whose contribution was incorrectly ignored in [12]. The correct massless six-dimensional open superstring vertex operator is $V = \int dz(\Pi^m A_m + \partial\theta^{\alpha j}A_{\alpha j} + d_{\alpha j}W^{\alpha j} + \frac{1}{2}(u\gamma^{mn}v)F_{mn})$ where F_{mn} is a superfield whose lowest component is the gluon field strength. Note that this vertex operator is annihilated on-shell by the 'harmonic' BRST-like operator $Q = \oint dz u^\alpha \widetilde{d}_\alpha$ and the central charge contribution from $v_\alpha \partial u^\alpha$ cancels the contribution from $p_{\alpha 2}\partial\theta^{\alpha 2}$ in the stress tensor to give a vanishing conformal anomaly.

In conformal gauge, the worldsheet action for the left-moving variables will be defined as

$$S = \int d^2z \left(\frac{1}{2}\partial x^m \overline{\partial} x_m + p_\alpha \overline{\partial}\theta^\alpha + \frac{1}{2} v^{ab}\overline{\partial} u_{ab} + \beta\overline{\partial}\gamma \right) \tag{2.5}$$

with the left-moving stress tensor

$$T = \frac{1}{2}\partial x^m \partial x_m + p_\alpha \partial\theta^\alpha + \frac{1}{2} v^{ab}\partial u_{ab} + \beta\partial\gamma \tag{2.6}$$

where (β, v^{ab}) are the conjugate momenta for (γ, u_{ab}). As desired, T has no conformal anomaly since the central contribution for the new degrees of freedom is 22, which cancels the central charge contribution from the x^m and $(\theta^\alpha, p_\alpha)$ variables.

In U(5) notation, the SO(10) Lorentz currents N^{mn} split into $(N, N_a^b, N^{ab}, N_{ab})$ which transform respectively as $(1_0, 24_0, 10_2, \overline{10}_{-2})$ representations. After fermionizing $\gamma = \eta e^\phi$ and $\beta = \partial\xi e^{-\phi}$ as in [7], N^{mn} will be defined as

$$\begin{aligned} N &= \frac{1}{\sqrt{5}}\left(u_{ab}v^{ab} + \frac{25}{4}\eta\xi + \frac{15}{4}\partial\phi \right), \qquad N_a^b = u_{ac}v^{bc} - \frac{1}{5}\delta_a^b u_{cd}v^{cd}, \\ N^{ab} &= v^{ab}, \qquad N_{ab} = 3\partial u_{ab} + u_{ac}u_{bd}v^{cd} + u_{ab}\left(\frac{5}{2}\eta\xi + \frac{3}{2}\partial\phi \right). \end{aligned} \tag{2.7}$$

Using the free-field OPE's,

$$\eta(y)\xi(z) \to (y-z)^{-1}, \quad \phi(y)\phi(z) \to -\log(y-z), \quad v^{ab}(y)u_{cd}(z) \to \delta_c^{[a}\delta_d^{b]}(y-z)^{-1}, \tag{2.8}$$

one can check that

$$\begin{aligned} N_a^b(y)N_c^d(z) &\to \frac{\delta_c^b N_a^d(z) - \delta_a^d N_c^b}{y-z} - 3\frac{\delta_a^d\delta_c^b - \frac{1}{5}\delta_a^b\delta_c^d}{(y-z)^2}, \qquad N(y)N(z) \to -\frac{3}{(y-z)^2}, \\ N_{ab}(y)N^{cd}(z) &\to \frac{-\delta_{[a}^{[c}N_{b]}^{d]}(z) - \frac{2}{\sqrt{5}}\delta_a^{[c}\delta_b^{d]}N(z)}{y-z} + 3\frac{\delta_a^{[c}\delta_b^{d]}}{(y-z)^2}, \end{aligned} \tag{2.9}$$

which correctly reproduces the OPE of (2.3).

Furthermore, $[\oint dz N^{mn}, \lambda^a] = \frac{1}{2}(\gamma^{mn}\lambda)^\alpha$ as can be easily shown by noting that $N^{mn} = \frac{1}{2}\omega\gamma^{mn}\lambda$ where ω_α is a spinor of the opposite chirality to λ^α with components[3]

$$\omega_+ = \xi e^{-\phi}\left(\eta\xi - \frac{1}{2}u_{ab}v^{ab} \right), \qquad \omega^{ab} = \xi e^{-\phi}v^{ab}, \qquad \omega_a = 0. \tag{2.10}$$

Note that

$$\omega_\alpha(y)\lambda^\beta(z) \to (y-z)^{-1}\delta_\alpha^\beta - \frac{1}{2}(y-z)^{-1}\gamma_m^{\beta+}\xi e^{-\phi}(\gamma^m\lambda)_\alpha \tag{2.11}$$

where the + in $\gamma_m^{\beta+}$ signifies the $1_{5/2}$ spinor component in the SU(5) notation of (2.4). The second term in the OPE of (2.11) is necessary for ω to have no singularity with $\lambda\gamma^m\lambda$, however, it does not contribute to the commutator $[\oint dz\omega\gamma^{mn}\lambda, \lambda]$ since $\lambda\gamma^m\gamma^{np}\lambda = 0$.

[3]Terms coming from normal-ordering ambiguities in $\omega\gamma^{mn}\lambda$ can be ignored since they only involve $\partial\phi + \eta\xi$ and ∂u_{ab}, which have no singularities with λ^α.

JHEP04(2000)018

So after introducing pure spinors, it is possible to obtain vanishing conformal anomaly and to relate the RNS gluon vertex operator with the proposal of Siegel in [4]. It will now be shown how these pure spinors can be used to define physical vertex operators and compute scattering amplitudes in a super-Poincaré covariant manner.

3. Physical vertex operators

Since the stress-tensor of (2.6) has vanishing central charge, one can require that physical vertex operators in unintegrated form are primary fields of dimension zero. However, this requirement is clearly insufficient since, for a massless vertex operator depending only on the zero modes of the worldsheet fields, it implies $\partial_m \partial^m \Phi(x, \theta, \lambda) = 0$ which has far more propagating fields than super-Yang-Mills. One therefore needs a further constraint on physical vertex operators, and using the intuition of [4], this constraint should be constructed from d_α of (2.1).

Using the pure spinor λ^α defined in terms of γ and u_{ab} as in (2.4), one can define a nilpotent BRST-like operator

$$Q = \oint dz \lambda^\alpha(z) d_\alpha(z)\,. \tag{3.1}$$

Defining ghost charge to be $q_{ghost} = \oint dz \gamma\beta$, Q carries ghost-number one. So it is natural to define physical vertex operators as states of ghost-number 1 in the cohomology of Q. Note that after Wick rotation, θ^α and λ^α are complex spinors, so the Hilbert space of states should be restricted to analytic functions of these variables.[4]

It will now be shown for the massless sector of the open superstring that this definition of physical states reproduces the desired super-Yang-Mills spectrum. Massless vertex operators of dimension zero can only depend on the worldsheet zero modes, so the most general such vertex operator of ghost number 1 is $U = \gamma Y(x, \theta, u_{ab})$ where Y is an analytic function of θ^α and u_{ab}. Since Q is Lorentz invariant (after including the contribution of N^{mn} (2.7) in the Lorentz generators), elements in its cohomology must transform Lorentz covariantly. But because of the non-linear nature of the Lorentz transformations generated by N^{mn}, the only finite-dimensional covariantly transforming object which is linear in γ is λ^α. So if the cohomology is restricted to finite-dimensional elements, the most general massless vertex operator of ghost number 1 is

$$U = \lambda^\alpha A_\alpha(x, \theta)\,, \tag{3.2}$$

where $A_\alpha(x, \theta)$ is a generic spinor function of x^m and θ^α.

[4] Although it is difficult to impose reality conditions on the states in euclidean space, this is not a problem for computing scattering amplitudes since it will be trivial to Wick-rotate the final result back to Minkowski space where the reality conditions are easily defined.

The constraint $QU = 0$ implies that $\lambda^\alpha \lambda^\beta D_\beta A_\alpha = 0$ where $D_\alpha = \frac{\partial}{\partial \theta^\alpha} + \gamma^m_{\alpha\beta}\theta^\beta \frac{\partial}{\partial x^m}$. Since $\lambda\gamma^m\lambda = 0$, this implies that $D_\alpha(\gamma^{mnpqr})^{\alpha\beta} A_\beta = 0$, which is the on-shell constraint for the spinor prepotential of super-Yang-Mills. Furthermore, the gauge transformation

$$\delta U = Q\Omega(x,\theta) = \lambda^\alpha D_\alpha \Omega(x,\theta) \tag{3.3}$$

reproduces the usual super-Yang-Mills gauge transformation $\delta A_\alpha = D_\alpha \Omega$ where $\Omega(x,\theta)$ is a generic scalar superfield. So the ghost number 1 cohomology of Q for the massless sector reproduces the desired super-Yang-Mills spectrum.

To compute scattering amplitudes, one also needs vertex operators in integrated form, i.e. integrals of dimension 1 primary fields. Normally, these are obtained from the unintegrated vertex operator by anti-commuting with $b(z)$. But in this formalism, there are no (b,c) ghosts, so it is presently unclear how to relate the two types of vertex operators. Nevertheless, one can define physical integrated vertex operators as elements in the BRST cohomology of ghost-number zero.

In the massless sector, there is an obvious candidate which is the dimension 1 vertex operator of (2.2) suitably modified to include the pure spinor contribution to the Lorentz current, i.e.

$$V = \Pi^m A_m + \partial\theta^\alpha A_\alpha + d_\alpha W^\alpha + N^{mn} F_{mn}\,, \tag{3.4}$$

where F_{mn} is the superfield whose lowest component is the gluon field strength. To show that $[Q, \int dz V] = 0$, first note that [4]

$$d_\alpha(y) \int dz v(z) \to \frac{1}{2}\int dz (y-z)^{-1} F_{mn}(z)(d(z)\gamma^{mn})_\alpha\,, \tag{3.5}$$

where $v(z) = \Pi^m A_m + \partial\theta^\alpha A_\alpha + d_\alpha W^\alpha$. Since $\lambda^\alpha(y) N^{mn}(z) \to \frac{1}{2}(y-z)^{-1}(\lambda(z)\gamma^{mn})^\alpha$,

$$\left[Q, \int dz V\right] = \int dz N^{mn} \lambda^\alpha D_\alpha F_{mn} = \int dz N^{mn} (\lambda\gamma_m \partial_n W)\,,$$

where W^α is the spinor field strength. But using $N^{mn} = \frac{1}{2}\omega\gamma^{mn}\lambda$ from (2.10),

$$N^{mn}(\lambda\gamma_m)_\alpha = \frac{1}{2}(w\gamma^{mn}\lambda)(\lambda\gamma_m)_\alpha = \frac{1}{2}(w_\beta\lambda^\beta)(\lambda\gamma^n)_\alpha$$

since $(\gamma^m\lambda)_\alpha(\gamma_m\lambda)_\beta = -\frac{1}{2}\gamma^m_{\alpha\beta}(\lambda\gamma_m\lambda) = 0$ from ten-dimensional γ-matrix identities. Finally, using the gluino equation of motion,

$$\left[Q, \int dz V\right] = \frac{1}{2}\int dz (w_\beta \lambda^\beta)(\lambda\gamma^n \partial_n W) = 0$$

so $\int dz V$ describes a physical integrated vertex operator. Note that the super-Yang-Mills gauge transformation $\delta A_M = \partial_M \Omega$ transforms V by the total derivative $\partial_z \Omega$, so $\int dz V$ is manifestly gauge-invariant.

4. Computation of scattering amplitudes

In this section, it will be shown how to compute tree-level open superstring scattering amplitudes in a manifestly super-Poincaré covariant manner. To compute N-point tree-level scattering amplitudes, one needs three vertex operators in unintegrated form and $N-3$ vertex operators in integrated form. Since only the massless vertex operators are known explicitly, only scattering of massless states will be considered here.

The two-dimensional correlation function which needs to be evaluated for computing tree-level scattering of N super-Yang-Mills multiplets is

$$\mathcal{A} = \langle U_1(z_1)U_2(z_2)U_3(z_3)\int dz_4 V_4(z_4)\cdots\int dz_N V_N(z_N)\rangle \tag{4.1}$$

where U_r is the dimension 0 vertex operator of (3.2), V_r is the dimension 1 vertex operator of (3.4), and the locations of (z_1, z_2, z_3) can be chosen arbitrarily because of $\mathrm{SL}(2,\mathbb{R})$ invariance.

The functional integral over the non-zero modes of the various worldsheet fields is completely straightforward using the free-field OPE's. For example, the dimension 1 worldsheet fields $(\partial x^m, \partial\theta^\alpha, d_\alpha, N^{mn})$ can be integrated out by contracting with other dimension 1 fields or with $(x^m, \theta^\alpha, \lambda^\alpha)$. Note that manifest Lorentz invariance is preserved by the contractions of N^{mn} because its only singular OPE's are $N^{mn}(y)\lambda^\alpha(z) \to \frac{1}{2}(y-z)^{-1}(\gamma^{mn}\lambda)^\alpha$ and (2.3).

However, the functional integral prescription for the zero modes of the worldsheet fields needs to be explained. Besides the zero modes of x^m (which are treated in the usual manner using conservation of momentum), there are the eleven bosonic zero modes of (γ, u_{ab}) and the sixteen fermionic zero modes of θ^α. After integrating out the non-zero modes, one gets an expression

$$\mathcal{A} = \int dz_4 \cdots \int dz_N \langle\lambda^\alpha\lambda^\beta\lambda^\gamma f_{\alpha\beta\gamma}(z_r, k_r, \theta)\rangle\,, \tag{4.2}$$

where only the zero modes of λ^α contribute and $f_{\alpha\beta\gamma}$ is a function which depends on $z_4, \ldots, z_N$, on the momenta k_r^m for $r = 1$ to N, and on the zero modes of θ^α.

The prescription for integration over the remaining worldsheet zero modes will be

$$\begin{aligned}\langle\lambda^\alpha\lambda^\beta\lambda^\gamma f_{\alpha\beta\gamma}(z_r, k_r, \theta)\rangle \equiv \int d\Omega\,(\overline{\lambda}_\delta\lambda^\delta)^{-3}\left(\frac{\partial}{\partial\theta}\gamma^{lmn}\frac{\partial}{\partial\theta}\right)\left(\overline{\lambda}\gamma_l\frac{\partial}{\partial\theta}\right)\left(\overline{\lambda}\gamma_m\frac{\partial}{\partial\theta}\right)\times\\ \times\left(\overline{\lambda}\gamma_n\frac{\partial}{\partial\theta}\right)\lambda^\alpha\lambda^\beta\lambda^\gamma f_{\alpha\beta\gamma}(z_r, k_r, \theta)\,,\end{aligned} \tag{4.3}$$

where $\overline{\lambda}_\alpha$ is the complex conjugate of λ^α in euclidean space and $d\Omega$ is an integration over the different possible orientions of λ^α. Although this prescription is defined in euclidean space, it is trivial to Wick-rotate the result back to Minkowski space using the fact that

$$\int d\Omega(\overline{\lambda}_\delta\lambda^\delta)^{-3}\lambda^\alpha\lambda^\beta\lambda^\gamma\overline{\lambda}_\rho\overline{\lambda}_\sigma\overline{\lambda}_\tau = T^{\alpha\beta\gamma}_{\rho\sigma\tau} \equiv \frac{1}{4032}\left[\delta^{(\alpha}_\rho\delta^\beta_\sigma\delta^{\gamma)}_\tau - \frac{1}{40}\gamma^{(\alpha\beta}_m\delta^{\gamma)}_{(\rho}\gamma^m_{\sigma\tau)}\right]. \tag{4.4}$$

Equation (4.4) can be derived using the fact that there is a unique covariantly transforming tensor $T^{\alpha\beta\gamma}_{\rho\sigma\tau}$ which is symmetrized with respect to its upper and lower indices and which satisfies $T^{\alpha\beta\gamma}_{\alpha\beta\gamma} = 1$ and $T^{\alpha\beta\gamma}_{\rho\sigma\tau}\gamma^m_{\alpha\beta} = T^{\alpha\beta\gamma}_{\rho\sigma\tau}\gamma^{\rho\sigma}_m = 0$.

So the amplitude of (4.2) can be written in SO(9, 1) Lorentz-covariant notation as

$$\mathcal{A} = T^{\alpha\beta\gamma}_{\rho\sigma\tau}\left(\frac{\partial}{\partial\theta}\gamma^{lmn}\frac{\partial}{\partial\theta}\right)\left(\gamma_l\frac{\partial}{\partial\theta}\right)^\rho\left(\gamma_m\frac{\partial}{\partial\theta}\right)^\sigma\left(\gamma_n\frac{\partial}{\partial\theta}\right)^\tau\int dz_4\cdots\int dz_N f_{\alpha\beta\gamma}(z_r, k_r, \theta)\,, \tag{4.5}$$

where $T^{\alpha\beta\gamma}_{\rho\sigma\tau}$ is defined in (4.4). By expanding $f_{\alpha\beta\gamma}(z_r, k_r, \theta)$ as a power series in θ^α, one can check that the prescription of (4.3) selects out the term

$$f_{\alpha\beta\gamma}(z_r, k_r, \theta) = \cdots + a(z_r, k_r)(\theta\gamma^{lmn}\theta)(\gamma_l\theta)_\alpha(\gamma_m\theta)_\beta(\gamma_n\theta)_\gamma + \cdots \tag{4.6}$$

in the power series, i.e. $\langle\lambda^\alpha\lambda^\beta\lambda^\gamma f_{\alpha\beta\gamma}(z_r, k_r, \theta)\rangle = a(z_r, k_r)$.

This prescription for integrating out the zero modes is reasonable since it is Lorentz invariant and since the eleven bosonic zero mode integrations are expected to cancel eleven of the sixteen fermionic zero mode integrations, leaving five zero modes of θ which are removed with five $\partial/\partial\theta$'s. Further evidence for this zero-mode prescription comes from the fact that it is gauge invariant and spacetime supersymmetric, as will now be shown.

To show that $\mathcal{A}$ is invariant under a gauge transformation

$$\delta U_1(z_1) = [\oint dz\lambda^\alpha d_\alpha, \Omega(z_1)]\,,$$

note that $\oint dz\lambda^\alpha d_\alpha$ commutes with U_r and $\int dz_r V_r$, so

$$\begin{aligned}\delta\mathcal{A} &= \left\langle\left[\oint dz\lambda^\alpha d_\alpha, \Omega(z_1)U_2(z_2)U_3(z_3)\int dz_4 V_4(z_4)\cdots\int dz_N V_N(z_N)\right]\right\rangle \\ &= \int dz_4\cdots\int dz_N\langle\lambda^\alpha\lambda^\beta\lambda^\gamma D_\alpha h_{\beta\gamma}(z_r, k_r, \theta)\rangle\end{aligned} \tag{4.7}$$

for some $h_{\beta\gamma}$ after integrating out the non-zero modes. Using the zero-mode prescription of (4.5),

$$\begin{aligned}\delta\mathcal{A} = T^{\alpha\beta\gamma}_{\rho\sigma\tau}&\left(\frac{\partial}{\partial\theta}\gamma^{lmn}\frac{\partial}{\partial\theta}\right)\left(\gamma_l\frac{\partial}{\partial\theta}\right)^\rho\left(\gamma_m\frac{\partial}{\partial\theta}\right)^\sigma\times \\ &\times\left(\gamma_n\frac{\partial}{\partial\theta}\right)^\tau\frac{\partial}{\partial\theta^\alpha}\int dz_4\cdots\int dz_N h_{\beta\gamma}(z_r, k_r, \theta)\,,\end{aligned} \tag{4.8}$$

where conservation of momentum has been used to replace D_α with $\partial/\partial\theta^\alpha$ in (4.8). But using anti-symmetry properties of $\partial/\partial\theta$, one can show that

$$T^{\alpha\beta\gamma}_{\rho\sigma\tau}\left(\frac{\partial}{\partial\theta}\gamma^{lmn}\frac{\partial}{\partial\theta}\right)\left(\gamma_l\frac{\partial}{\partial\theta}\right)^\rho\left(\gamma_m\frac{\partial}{\partial\theta}\right)^\sigma\left(\gamma_n\frac{\partial}{\partial\theta}\right)^\tau\frac{\partial}{\partial\theta^\alpha}=0\,,$$

so $\delta\mathcal{A}=0$.

It will now be shown that the prescription of (4.5) is invariant under spacetime supersymmetry transformations, implying that the amplitudes are SO(9,1) super-Poincaré invariant. Under a spacetime supersymmetry transformation with global parameter ϵ^κ, the term $a(z_r,k_r)$ of (4.6) transforms as $\delta a(z_r,k_r)=\epsilon^\kappa\xi_\kappa(z_r,k_r)$ where ξ_κ appears in the power series for $f_{\alpha\beta\gamma}$ as

$$f_{\alpha\beta\gamma}(z_r,k_r,\theta)=\cdots+(\theta\gamma^{lmn}\theta)(\gamma_l\theta)_\alpha(\gamma_m\theta)_\beta(\gamma_n\theta)_\gamma[a(z_r,k_r)+\theta^\kappa\xi_\kappa(z_r,k_r)]+\cdots \quad (4.9)$$

But $\int dz_4\cdots\int dz_N\lambda^\alpha\lambda^\beta\lambda^\gamma f_{\alpha\beta\gamma}$ comes from vertex operators which commute with $\oint dz\lambda^\delta d_\delta$, so it must satisfy the constraint

$$\int dz_4\cdots\int dz_N\lambda^\alpha\lambda^\beta\lambda^\gamma\lambda^\delta D_\delta f_{\alpha\beta\gamma}=0 \quad (4.10)$$

for any pure spinor λ^α. Plugging (4.9) into (4.10) and using

$$(\lambda\gamma^{lmn}\theta)(\lambda\gamma_l\theta)(\lambda\gamma_m\theta)(\lambda\gamma_n\theta)=0\,,$$

one finds that

$$\int dz_4\cdots\int dz_N(\theta\gamma^{lmn}\theta)(\lambda\gamma_l\theta)(\lambda\gamma_m\theta)(\lambda\gamma_n\theta)\lambda^\kappa\xi_\kappa(z_r,k_r) \quad (4.11)$$

must vanish for any pure spinor λ^α. But this is only possible if

$$\int dz_4\cdots\int dz_N\xi_\kappa(z_r,k_r)=0\,,$$

implying that

$$\delta\mathcal{A}=\int dz_4\cdots\int dz_N\delta a(z_r,k_r)=0 \quad (4.12)$$

so the amplitude prescription of (4.5) is spacetime supersymmetric.

5. Superstring action in a curved background

In this section, the massless integrated vertex operator for the closed superstring will be used to construct a quantizable action for the superstring in a curved background. As a special case, a quantizable action will be constructed for the type-IIB superstring in an $AdS_5\times S^5$ background with Ramond-Ramond flux.

JHEP04(2000)018

In bosonic string theory and in the Neveu-Schwarz sector of superstring theory, the action in a curved background (ignoring the Fradkin-Tseytlin term for dilaton coupling) can be constructed by 'covariantizing' the massless closed string vertex operator with respect to target-space reparameterization invariance. As in [13] and [12], this procedure can also be used here after constructing the massless closed string vertex operator from the 'left-right' product of two massless open string vertex operators of (3.4).

To do this, one first needs to introduce right-moving analogs of the worldsheet fields described in (2.5). The complete worldsheet action for the type-II superstring in a flat background in conformal gauge is

$$S = \int d^2z \left(\frac{1}{2}\partial x^m \overline{\partial} x_m + p_\alpha \overline{\partial}\theta^\alpha + \widehat{p}_{\widehat{\alpha}}\partial\widehat{\theta}^{\widehat{\alpha}} + \frac{1}{2}(v^{ab}\overline{\partial}u_{ab} + \widehat{v}^{ab}\partial\widehat{u}_{ab}) + \beta\overline{\partial}\gamma + \widehat{\beta}\partial\widehat{\gamma}\right), \tag{5.1}$$

where $\widehat{\lambda}^{\widehat{\alpha}}$ is constructed from $\widehat{\gamma}$ and $\widehat{u}_{ab}$ in a manner similar to (2.4). Note that $\widehat{\theta}^{\widehat{\alpha}}$ and $\widehat{\lambda}^{\widehat{\alpha}}$ are independent of θ^α and λ^α, and are not related by complex conjugation. For the type-IIA superstring, the hatted spinor index has the opposite chirality to the unhatted spinor index while, for the type-IIB superstring, the hatted spinor index has the same chirality as the unhatted spinor index.

The action for the type-II superstring in a curved background obtained by 'covariantizing' the massless closed superstring vertex operator with respect to target-space super-reparameterization invariance is[5]

$$\begin{aligned} S = \int d^2z \Big[& \frac{1}{2}(G_{MN} + B_{MN})\partial y^M \overline{\partial} y^N + (d_\alpha + N_\alpha^\beta D_\beta)E_M^\alpha \overline{\partial} y^M + \\ & + (\widehat{d}_{\widehat{\alpha}} + \widehat{N}_{\widehat{\alpha}}^{\widehat{\beta}}\widehat{D}_{\widehat{\beta}})E_M^{\widehat{\alpha}}\partial y^M + (d_\alpha + N_\alpha^\beta D_\beta)(\widehat{d}_{\widehat{\gamma}} + \widehat{N}_{\widehat{\gamma}}^{\widehat{\delta}}\widehat{D}_{\widehat{\delta}})P^{\alpha\widehat{\gamma}} + \\ & + \frac{1}{2}(v^{ab}\overline{\partial}u_{ab} + \widehat{v}^{ab}\partial\widehat{u}_{ab}) + \beta\overline{\partial}\gamma + \widehat{\beta}\partial\widehat{\gamma}\Big], \end{aligned} \tag{5.2}$$

where y^M parameterizes the curved superspace background, E_M^α and $E_M^{\widehat{\alpha}}$ are the spinor parts of the super-vierbein E_M^A, $N_\alpha^\beta \equiv N^{mn}(\gamma_{mn})_\alpha^\beta$ and $\widehat{N}_{\widehat{\alpha}}^{\widehat{\beta}} \equiv \widehat{N}^{mn}(\gamma_{mn})_{\widehat{\alpha}}^{\widehat{\beta}}$ with N_{mn} being defined by (2.7) and $\widehat{N}_{mn}$ being defined similarly in terms of the hatted variables, and $P^{\alpha\widehat{\gamma}}$ is the superfield whose lowest components are the bispinor Ramond-Ramond field strengths. The operators D_α and $\widehat{D}_{\widehat{\alpha}}$ in (5.2) are understood to act on the superfield to their left, e.g. on E_M^α, $E_M^{\widehat{\alpha}}$ or $P^{\alpha\widehat{\gamma}}$.

Note that the first line of (5.2) is identical to the Green-Schwarz action in a curved background, however, the second and third lines are crucial for quantization since they provide an invertible propagator for θ^α and $\widehat{\theta}^{\widehat{\alpha}}$. Furthermore, since there

[5] In the action for the superstring in a curved six-dimensional background, there are terms coming from the bosonic ghosts (u^α, v_α) described in footnote 2 which were incorrectly omitted from the action of [12]. The correct action should have terms of the type $\int d^2z(d_{\alpha j}+u_\alpha v^\beta D_{\beta j})E_M^{\alpha j}\overline{\partial}Y^M+\cdots)$, as well as a kinetic action for the (u^α, v_α) and $(\widehat{u}^{\widehat{\alpha}}, \widehat{v}_{\widehat{\alpha}})$ ghosts.

is no fermionic κ-symmetry which needs to be preserved, there is no problem with adding a Fradkin-Tseytlin term to (5.2) of the type $\alpha' \int d^2 z \Phi(x,\theta,\widehat{\theta}) R$ where R is the worldsheet curvature and Φ is a scalar superfield whose lowest component is the dilaton.

When the background superfields satisfy their effective low-energy equations of motion, the action of (5.2) together with the Fradkin-Tseytlin term is expected to be conformally invariant where the left-moving stress tensor is

$$T = \frac{1}{2} G_{MN} \partial y^M \partial y^N + (d_\alpha + N_\alpha^\beta D_\beta) E_M^\alpha \partial y^M + \frac{1}{2} v^{ab} \partial u_{ab} + \beta \partial \gamma + \alpha' \partial^2 \Phi \,. \tag{5.3}$$

Furthermore, the current $\lambda^\alpha d_\alpha$ is expected to be holomorphic and nilpotent when the background superfields are on-shell.

5.1 Superstring action in $AdS_5 \times S^5$ background

In this subsection, the action of (5.2) will be explicitly constructed for the special case of the type-IIB superstring in an $AdS_5 \times S^5$ background with Ramond-Ramond flux. As discussed in [10], this background can be conveniently described by a coset supergroup g taking values in $\mathrm{PSU}(2,2|4)/\,\mathrm{SO}(4,1)\times \mathrm{SO}(5)$ where the super-vierbein E_M^A satisfies

$$E_M^A dy^M = (g^{-1} dg)^A \tag{5.4}$$

and $A = (c, \alpha, \widehat{\alpha})$ ranges over the 10 bosonic and 32 fermionic entries in the Lie-algebra valued matrix $g^{-1}dg$. Furthermore, as discussed in [14], the only non-zero components of $B_{AB} = E_A^M E_B^N B_{MN}$ and $P^{\alpha\widehat{\beta}}$ are

$$B_{\alpha\widehat{\beta}} = B_{\widehat{\beta}\alpha} = -\frac{1}{2ng_s} \delta_{\alpha\widehat{\beta}}, \quad P^{\alpha\widehat{\beta}} = ng_s \delta^{\alpha\widehat{\beta}}, \tag{5.5}$$

where n is the value of the Ramond-Ramond flux, g_s is the string coupling constant, and $\delta_{\alpha\widehat{\beta}} = (\gamma^{01234})_{\alpha\widehat{\beta}}$ with 01234 being the directions of AdS_5.

Plugging these background superfields into the action of (5.2), one finds

$$\begin{aligned} S_{AdS} = \int d^2 z \Bigg[& \frac{1}{2} \eta_{cd} (g^{-1}\partial g)^c (g^{-1}\overline{\partial} g)^d - \\ & - \frac{1}{4ng_s} \delta_{\alpha\widehat{\beta}} \left[(g^{-1}\partial g)^\alpha (g^{-1}\overline{\partial} g)^{\widehat{\beta}} - (g^{-1}\overline{\partial} g)^\alpha (g^{-1}\partial g)^{\widehat{\beta}} \right] + \\ & + d_\alpha (g^{-1}\overline{\partial} g)^\alpha + \widehat{d}_{\widehat{\alpha}} (g^{-1}\partial g)^{\widehat{\alpha}} + ng_s \delta^{\alpha\widehat{\beta}} d_\alpha \widehat{d}_{\widehat{\beta}} + N_{cd} (g^{-1}\overline{\partial} g)^{cd} + \\ & + \widehat{N}_{cd} (g^{-1}\partial g)^{cd} + \frac{1}{2} (v^{ab} \overline{\partial} u_{ab} + \widehat{v}^{ab} \partial \widehat{u}_{ab}) + \beta \overline{\partial} \gamma + \widehat{\beta} \partial \widehat{\gamma} \Bigg] \,, \end{aligned} \tag{5.6}$$

where $(g^{-1}dg)^{cd}$ are the $\mathrm{SO}(4,1) \times \mathrm{SO}(5)$ coset elements of $g^{-1}dg$. This action is $\mathrm{PSU}(2,2|4)$-invariant since under $g \to Mg\Omega$ with M a global $\mathrm{SU}(2,2|4)$ matrix and

Ω a local SO(4,1) $\times$ SO(5) matrix assumed to be close to the identity,

$$\delta(g^{-1}dg)^A = ([g^{-1}dg, \Omega])^A, \qquad \delta(g^{-1}dg)^{cd} = ([g^{-1}dg, \Omega])^{cd} + (d\Omega)^{cd},$$
$$\delta\left(\frac{1}{2}\left(v^{ab}\overline{\partial}u_{ab} + \widehat{v}^{ab}\partial\widehat{u}_{ab}\right) + \beta\overline{\partial}\gamma + \widehat{\beta}\partial\widehat{\gamma}\right) = -N_{cd}(\overline{\partial}\Omega)^{cd} - \widehat{N}_{cd}(\partial\Omega)^{cd}, \quad (5.7)$$

where $(d_\alpha, \widehat{d}_{\widehat{\alpha}})$ and $(\lambda^\alpha, \widehat{\lambda}^{\widehat{\alpha}})$ are defined to transform as Lorentz-covariant spinors under the local SO(4,1) $\times$ SO(5) transformation.

Because of the $ng_s\delta^{\alpha\widehat{\beta}}d_\alpha\widehat{d}_{\widehat{\beta}}$ term, d_α and $\widehat{d}_{\widehat{\alpha}}$ are auxiliary fields which can be integrated out as was done in [15]. Their auxiliary equations of motion are

$$d_\alpha = \frac{1}{ng_s}\delta_{\alpha\widehat{\beta}}(g^{-1}\partial g)^{\widehat{\beta}}, \qquad \widehat{d}_{\widehat{\beta}} = -\frac{1}{ng_s}\delta_{\alpha\widehat{\beta}}(g^{-1}\partial g)^{\alpha}, \quad (5.8)$$

which can be substituted into (5.6) to give

$$S_{AdS} = \int d^2z\left[\frac{1}{2}\eta_{cd}(g^{-1}\partial g)^c(g^{-1}\overline{\partial}g)^d - \frac{3}{4ng_s}\delta_{\alpha\widehat{\beta}}(g^{-1}\overline{\partial}g)^\alpha(g^{-1}\partial g)^{\widehat{\beta}}\right.$$
$$- \frac{1}{4ng_s}\delta_{\alpha\widehat{\beta}}(g^{-1}\partial g)^\alpha(g^{-1}\overline{\partial}g)^{\widehat{\beta}} + N_{cd}(g^{-1}\overline{\partial}g)^{cd} + \widehat{N}_{cd}(g^{-1}\partial g)^{cd} +$$
$$\left.+\frac{1}{2}(v^{ab}\overline{\partial}u_{ab} + \widehat{v}^{ab}\partial\widehat{u}_{ab}) + \beta\overline{\partial}\gamma + \widehat{\beta}\partial\widehat{\gamma}\right]. \quad (5.9)$$

Finally, one can perform the rescaling

$$E_M^c \to (ng_s)^{-1}, \qquad E_M^\alpha \to (ng_s)^{-1/2}, \qquad E_M^{\widehat{\alpha}} \to (ng_s)^{-1/2}, \quad (5.10)$$

to obtain the action[6]

$$S_{AdS} = \frac{1}{n^2g_s^2}\int d^2z\left[\frac{1}{2}\eta_{cd}(g^{-1}\partial g)^c(g^{-1}\overline{\partial}g)^d - \right. \quad (5.11)$$
$$\left.- \frac{3}{4}\delta_{\alpha\widehat{\beta}}(g^{-1}\overline{\partial}g)^\alpha(g^{-1}\partial g)^{\widehat{\beta}} - \frac{1}{4}\delta_{\alpha\widehat{\beta}}(g^{-1}\partial g)^\alpha(g^{-1}\overline{\partial}g)^{\widehat{\beta}}\right] +$$
$$+\int d^2z\left[N_{cd}(g^{-1}\overline{\partial}g)^{cd} + \widehat{N}_{cd}(g^{-1}\partial g)^{cd} + \frac{1}{2}(v^{ab}\overline{\partial}u_{ab} + \widehat{v}^{ab}\partial\widehat{u}_{ab}) + \beta\overline{\partial}\gamma + \widehat{\beta}\partial\widehat{\gamma}\right].$$

Except for the third line of (5.11) involving the pure spinor fields, this is precisely the action which was proposed in [14]. Note that unlike the action proposed in [10], the action of (5.11) is straightforward to quantize and sigma model loop computations can be performed in the manner of [14].

[6] Two quantizable actions have been proposed for the superstring in an $AdS_3 \times S^3$ background with Ramond-Ramond flux [15, 12]. One of the actions includes eight left and right-moving θ coordinates and is based on the coset supergroup PSU(1,1|2) $\times$ PSU(2|2)/ SU(2) $\times$ SU(2) [12]. For the reasons discussed in footnotes 2 and 5, this action requires the addition of terms involving the bosonic ghosts (u^α, v_α) and $(\widehat{u}^{\widehat{\alpha}}, \widehat{v}_{\widehat{\alpha}})$. The other action includes four left and right-moving θ coordinates and is based on the supergroup PSU(2|2) [15]. This action does not involve the 'harmonic' constraints discussed in [12] and therefore does not require the addition of any terms involving bosonic ghosts.

JHEP04(2000)018

6. Concluding remarks

In this paper, a new formalism has been presented for covariantly quantizing the superstring. For the first time, vertex operators have been constructed and scattering amplitudes have been computed in a manifestly ten-dimensional super-Poincaré invariant manner. A quantizable action has been proposed for the superstring in any curved background, including the $AdS_5 \times S^5$ background with Ramond-Ramond flux.

There are various possible generalizations of the new formalism which should be possible. For example, one should be able to generalize the massless vertex operators to massive vertex operators and generalize the tree-level amplitude prescription to a multiloop amplitude prescription. One should also be able to construct physical vertex operators in the $AdS_5 \times S^5$ background in a manner similar to the construction of $AdS_3 \times S^3$ vertex operators in [16].

A more ambitious application would be to use the new formalism to construct a second-quantized superstring field theory. Note that the physical state condition $QV = 0$ is easily generalized to the non-linear equation of motion $QV + V \wedge V = 0$ where the $\wedge$-product is defined as in [17]. However, since the scattering amplitudes of section 4 are only spacetime-supersymmetric when the states satisfy the on-shell condition $QV = 0$, it is unlikely that the action which produces this equation of motion will be manifestly spacetime-supersymmetric.

Perhaps the most important unresolved issue is to prove the equivalence between this formalism and the RNS formalism. In section 2, preliminary evidence for this equivalence came from comparing the gluon vertex operator in the two formalisms. Further evidence for this equivalence will now be shown by considering the pure spinor formalism in the "U(5) gauge" defined by setting $u_{ab} = \theta_{ab} = 0$.

In this U(5) gauge, the operator $\oint dz \lambda^\alpha d_\alpha$ reduces to $\oint dz \gamma d_+$. If one compares with the U(5)-invariant formalism of [8], it is natural to identify[7] $\gamma = e^{-\frac{1}{2}(\rho - i\sigma)}$, or using the field redefinition of [8] to RNS worldsheet variables, $\gamma = c e^{-\frac{3}{2}\phi} \Sigma^+$ where Σ^α is the RNS spin field. Since $d_+ = b\eta e^{\frac{3}{2}\phi} \Sigma_+$ using this same field redefinition, one finds $\oint dz \lambda^\alpha d_\alpha = \oint dz \eta$ in the U(5) gauge. So the restriction that $\oint dz \lambda^\alpha d_\alpha$ commutes with physical states maps in this gauge to the restriction that physical states should be independent of the ξ zero mode.

Note that the unintegrated vertex operator for super-Yang-Mills, $U = \lambda^\alpha A_\alpha$, reduces in the U(5) gauge to

$$U = \gamma \left(A_+ + \theta^a A_a + \frac{1}{2} \theta^a \theta^b W_{ab} + \frac{1}{12} \epsilon_{abcde} \theta^a \theta^b \theta^c F^{de} + \frac{1}{24} \epsilon_{abcde} \theta^a \theta^b \theta^c \theta^d \partial^e W^+ \right), \tag{6.1}$$

where A_a is the $\overline{5}_{-1}$ component of the gluon gauge field and F^{ab} is the 10_2 component of the gluon field strength. Using the field redefinition of [8] to write U in terms of

[7] In the following discussion, γ always refers to λ^+ and not to the RNS bosonic ghost.

RNS variables where $\theta^a = e^{\frac{1}{2}\phi}\Sigma^a$, one finds

$$U = c\left(e^{-\frac{3}{2}\phi}\Sigma^+ A_+ + e^{-\phi}\psi^a A_a + \frac{1}{2}e^{-\frac{1}{2}\phi}\Sigma^{ab}W_{ab} + \frac{1}{2}\psi_d\psi_e F^{de} + e^{\frac{1}{2}\phi} : \Sigma^-\psi_e : \partial^e W^+\right), \tag{6.2}$$

where $:\Sigma^-\psi_e:$ signifies $\lim_{y\to z}(y-z)^{-\frac{1}{2}}\Sigma^-(y)\psi_e(z)$. Except for the term proportional to $\partial^e W^+$, all of the terms in U can be recognized as pieces of the RNS gluon and gluino vertex operators in various pictures.

Furthermore, if only the contribution from $u_{ab} = 0$ is included, the integration prescription of 4.3 implies that

$$\langle\lambda^\alpha\lambda^\beta\lambda^\gamma f_{\alpha\beta\gamma}(z_r, k_r, \theta)\rangle \equiv \left(\frac{\partial}{\partial\theta}\right)^5 f_{+++}(z_r, k_r, \theta), \tag{6.3}$$

i.e. $\langle\gamma^3(\theta)^5\rangle = 1$ where $(\theta)^5 = (5!)^{-1}\epsilon_{abcde}\theta^a\theta^b\theta^c\theta^d\theta^e$. But under the field redefinition of [8], $\gamma^3(\theta)^5$ maps to $c\partial c\partial^2 c e^{-2\phi}$, as expected from $\langle\gamma^3(\theta)^5\rangle = 1$.

So it appears reasonable that the U(5) gauge-fixed version of the new formalism is equivalent to the RNS formalism with a fixed choice of picture for the vertex operators. Removing the gauge-fixing condition, i.e. integrating over the u_{ab} and θ_{ab} variables using the prescription of (4.3), might then be equivalent to summing over the different possible pictures in the RNS formalism. It would be very useful to be able to prove such an equivalence.

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JHEP04(2000)018